**Table 1 Summary of results**

| | $Re=45374$ | | $Re=11345$ | | $Re=2269$ | |
|---|---|---|---|---|---|---|
| | $\Delta t/\Delta t_{CFL}$ | $N$ | $\Delta t/\Delta t_{CFL}$ | $N$ | $\Delta t/\Delta t_{CFL}$ | $N$ |
| MacCormack[a] | 0.9 | 338 | 0.9 | 546 | 0.5 | 896 |
| Modified hopscotch[a] | 1.0 | 565 | 0.8 | 514 | 0.3 | 1193 |
| Brailovskaya[a] | 1.1 | 323 | 1.0 | 485 | 0.4 | 1110 |
| Modified DuFort-Frankel[a] | 0.5 | 990 | 0.4 | 1350 | ... | ... |
| Stetter | 2.1 | 181 | 2.1 | 197 | 0.4 | 995 |

[a] Taken from Ref. 2.

number cases, Stetter's method reaches steady state in many fewer time steps than the other four methods. For the lowest Reynolds number case, which is viscous dominated, all of the methods take approximately the same number of steps. It should be noted that Stetter's method requires more work per time step (three evaluations of the space derivatives as opposed to one or two for the other methods); although the CPU time in the present study was not accurately determined, it does appear that the modified hopscotch has a slightly better CPU time. However, there are applications such as Ref. 4, where auxiliary transport calculations require more CPU time than the fluid mechanics, and in such a situation, even though the auxiliary calculations are not made at every time step, the total CPU time for a given problem is reduced by having the fluid mechanics converge faster. Thus, Stetter's method with its ability to use larger marching steps, should find application in such problems.

## Concluding Remarks

Stetter's three-step predictor-corrector technique has been used to solve the compressible Navier-Stokes equations for quasi-one-dimensional flow in a converging-diverging nozzle. For the nonviscous dominated case, Stetter's method was clearly superior to the four other currently popular methods in attaining the steady-state solution in the fewest steps. Time steps of slightly greater than two times the CFL limit were obtainable with Stetter's method.

## References

[1] Peyret, R. and Viviand, H., "Computation of Viscous Compressible Flows Based on the Navier-Stokes Equations," AGARDograph No. 212, Sept. 1975.

[2] Holst, T. L., "The Relative Merits of Several Numerical Techniques for Solving the Compressible Naiver-Stokes Equations," *13th Annual Meeting of the Society of Engineering Science*, Sponsored by JIAFS, Hampton, Va., Nov. 1-3, 1976, NASA CP-2001, Vol. II, p. 1467.

[3] Stetter, H. J., "Improved Absolute Stability of Predictor-Corrector Schemes," *Computing*, Vol. 3, 1968, p. 286.

[4] Sutton, K., "Characteristics of Coupled Nongray Radiating Gas Flows with Ablation Product Effects About Blunt Bodies During Planetary Entries," Ph.D. Thesis, Dept. of Mechanical Engineering, North Carolina State Univ., 1973.

# Integral Equation Formulation for Transonic Flow Past Lifting Wings

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## Introduction

THE integral equation method for calculating steady inviscid transonic flows based on small perturbation theory is well known. In general it involves a small amount of numerical work. For the three-dimensional case an integral equation formulation originates from Klunker[1] by the application of Green's theorem. It also gives the expression for the far-field potential for three-dimensional wings at transonic speeds. By means of the small disturbance shock jump conditions, Klunker showed that the surface integral term over the surface of the shock discontinuity does not make any contribution to the potential. Note that Ferrari and Tricomi[2] made a normal shock assumption in the integral equation formulation for two-dimensional nonlifting flows. There, the integral equation was formulated in terms of the velocity component instead of the velocity potential, as was done by Klunker. It is apparent from the work of Klunker that the normal shock assumption, which is one of the drawbacks of the integral equation method, is not necessary if we formulate the integral equation in terms of the velocity potential and subsequently differentiate it to get the same integral equation for the velocity components.[1]

On the other hand, the analytical solution to the transonic small perturbation equation in reduced coordinates was expressed as a nonlinear integral equation by Nörstrud,[3] with the reduced perturbation velocity potential as the unknown function. The purpose of the present Note is to study critically Nörstrud's formulation for three-dimensional transonic lifting wings and to indicate an alternative way of deriving them.

Received March 15, 1978; revision received May 10, 1978.

Index categories: Subsonic Flow; Transonic Flow.
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## An Alternative Deduction of the Integral Equation Formulation

Using the same notations defined by Niyogi,[4] the integral equation formulated by Nörstrud[3] for reduced $U$ velocity components may be written as follows

$$U(X,\pm 0,Z)=U_P(X,\pm 0,Z)\pm \tfrac{1}{2}[\Delta U(X,Z)-\Delta U_P(X,Z)] + \lim_{Y\to 0^\pm}\frac{\partial I(X,Y,Z;U)}{\partial X} \tag{1}$$

where $U_P(X,Y,Z)=(\partial/\partial X)\Phi_P(X,Y,Z)$; $\Phi_P$ the solution of Laplace equation

$$\Phi_{XX}+\Phi_{YY}+\Phi_{ZZ}=0 \tag{2}$$

satisfies the same boundary condition as $U(X,Y,Z)$. $I(X,Y,Z;U)$ represents the volume integral

$$I(X,Y,Z;U)=\frac{-1}{4\pi}\iiint_{-\infty}^{\infty}\Phi_\xi(\xi,\eta,\zeta)\Phi_{\xi\xi}(\xi,\eta,\zeta)\frac{1}{R}\,d\xi d\eta d\zeta \tag{3}$$

It has been shown in Ref. 4 that the two-dimensional equations corresponding to Eq. (1) are not independent. Proceeding in a similar way, the same conclusion can be drawn for the present three-dimensional case also. Alternatively, following Nörstrud,[5] if the symmetric and antisymmetric parts of $U(X,Y,Z)$ and $U_P(X,Y,Z)$ are defined by $U^+$, $U^-$, and $U_p^+$, $U_p^-$, respectively, then Eqs. (1) becomes

$$U^+ + U^- = U_p^+ + U_p^- + U^- - U_p^- + \lim_{Y\to 0^+}(\partial I/\partial X)$$

or

$$U^+ = U_p^+ + \lim_{Y\to 0^+}(\partial I/\partial X) \tag{4}$$

and

$$U^+ - U^- = U_p^+ - U_p^- - U^- + U_p^- + \lim_{Y\to 0^-}(\partial I/\partial X)$$

or

$$U^+ = U_p^+ + \lim_{y\to 0^-}(\partial I/\partial X) \tag{5}$$

The relation

$$\lim_{Y\to 0^+} (\partial I/\partial X) = \lim_{Y\to 0^-} (\partial I/\partial X)$$

comes from Eqs. (4) and (5) and can also be proved following Refs. 4 and 6. Note that Eqs. (4) and (5) are identical and represent a single equation for the symmetric part of the velocity only. The second requisite independent equation may be obtained by differentiating the integral equation for $\Phi(X,Y,Z)$ with respect to $Y$ along with the tangency boundary condition at the wing surface. The problem is thus reduced to the solution of the coupled pair of integral equations, and an iteration is necessary to solve them.

An alternative formulation of the integral equations (1) can be obtained as follows. By the application of Green's theorem, the three-dimensional transonic equation for the reduced perturbation potential $\Phi$ may be converted to the following integral equation, as shown in detail by Klunker.[1]

$$\Phi(X,Y,Z) = \frac{Y}{4\pi}\int_S \frac{\Delta U}{(Z-\zeta)^2+Y^2}\left(1+\frac{X-\xi}{R}\right)\mathrm{d}S - \int_S \Psi\Delta V \mathrm{d}S + \int_{\bar V} U^2\Psi_\xi \mathrm{d}\bar V \tag{6}$$

where

$$\Psi = 1/4\pi R \tag{7}$$

Here, $S$ represents the wing planform area and $\bar V$ the total volume enclosed by the surface at infinity, the surface surrounding the singular point, the surface around any shock discontinuity, the wing surface, and the trailing vortex sheet.

The harmonic solution $\Phi_P$ for linearized subsonic theory can be written as

$$\Phi_P = -\int_S \Psi\Delta V_P \mathrm{d}S + \frac{Y}{4\pi}\int_S \frac{\Delta U_P}{Y^2+(Z-\zeta)^2}\left(1+\frac{X-\xi}{R}\right)\mathrm{d}S \tag{8}$$

$\Phi_P$ satisfies the same boundary conditions as $\Phi$ corresponding to the nonlinear theory. So, according to the thin airfoil theory

$$\Delta\Phi_Y(X,0,Z) = \Delta\Phi_{PY}(X,0,Z), \quad 0\le X\le 1 \tag{9}$$

Since the integral over $S$ in Eqs. (6) and (8) means integration over only the wing planform, the second term on the right-hand side of Eq. (6) equals to the first term of the right-hand side of Eq. (8), in view of Eq. (9).

The subtracting Eq. (8) from Eq. (6) follows

$$\Phi(X,Y,Z) = \Phi_P(X,Y,Z) + \frac{Y}{4\pi}\int_S \frac{\Delta U - \Delta U_P}{Y^2-(Z-\zeta)^2} \times \left(1+\frac{X-\xi}{R}\right)\mathrm{d}S + I'(X,Y,Z;U) \tag{10}$$

where

$$I'(X,Y,Z;U) = \int_{\bar V} U^2\Psi_\xi \mathrm{d}\bar V$$

Differentiating Eq. (10) with respect to $X$ gives

$$U(X,Y,Z) = U_P(X,Y,Z) + \frac{Y}{4\pi}\int_S \frac{\Delta U(\xi,0,\zeta) - \Delta U_P(\xi,0,\zeta)}{[(X-\xi)^2+Y^2+(Z-\zeta)^2]^{3/2}}\mathrm{d}S + \frac{\partial I'}{\partial X} \tag{11}$$

Now, taking limits as $Y\to \pm 0$ and considering the second term on the right-hand side of Eq. (11) as the well-known Poisson's integral, it follows for continuous flow that

$$U(X,\pm 0,Z) = U_P(X,\pm 0,Z) \pm \tfrac{1}{2}[\Delta U(X,0,Z) - \Delta U_P(X,0,Z)] + \lim_{Y\to\pm 0}(\partial I'/\partial X) \tag{12}$$

If there is no shock discontinuity, it can be shown[1] by integrating $I(X,Y,Z,U)$ by parts in the $X$ direction that $I(X,Y,Z,U) = I'(X,Y,Z,U)$, and so the Nörstrud's integral Eqs. (1) result. If there is a shock discontinuity, then integrating by parts in the $X$ direction $I(X,Y,Z,U)$ gives

$$I(X,Y,Z,U) = -\int_{\text{shock}} \Psi[U^2]\cos\sigma_1 \mathrm{d}S + I'(X,Y,Z;U) \tag{13}$$

where the upstream unit normal to the shock surface has been taken as

$$\bar n = i\cos\sigma_1 + j\cos\sigma_2 + K\cos\sigma_3$$

Now, the first term on the right-hand side of Eq. (13) along with the integral terms over the shock surface which appeared in the course of formulating Eq. (6) give no contribution because of the small disturbance approximation of shock jump conditions.[1] Thus, Eq. (11) as well as Eq. (12) is valid, for the piecewise continuous function $\Delta U(X,Y,Z)$ on the boundary $Y=0$, even for the case when shock discontinuity appears.

### Acknowledgment

The author takes this opportunity to thank P. Niyogi of Jadavpur University and B. C. Basu for their continuous encouragement and valuable suggestions while preparing this Note.

### References

[1] Klunker, E. B., "Contribution to Methods for Calculating the Flow about Thin Lifting Wings at Transonic Speeds—Analytic Expressions for the Far Field," NASA TND-6530, Nov. 1971.

[2] Ferrari, C. and Tricomi, F. G., *Transonic Aerodynamics*, Academic Press, New York, 1968, Chap. VI, Sec. 10.1.

[3] Nörstrud, H., "High Speed Flow Past Wings," NASA CR-2246, April 1973.

[4] Niyogi, P., "Transonic Integral Equation for Lifting Profiles and Wings," *AIAA Journal*, Vol. 16, Jan. 1978, pp. 92-94.

[5] Nörstrud, H., "Numerische Lösungen für Schallnahe Strömungen um Ebene Profile," *Zeitschrift für Flugwissenschaften*, Vol. 18, Sept. 1970, pp. 149-157.

[6] Epstein, B., *Partial Differential Equations*, McGraw Hill Inc., New York, 1962, Chap. 6, Article 12.

# Calculation of Two-Dimensional Turbulent Boundary Layers

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## Introduction

In this Note, the boundary layer over a flat plate is calculated with a three-equation model of turbulence which makes use of transport equations for the kinetic energy $k$, the

Received Dec. 27, 1977; revision received April 20, 1978. 
Index categories: Boundary Layers and Convective Heat Transfer—Turbulent; Computational Methods.

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